

Position-aware Structure Learning for Graph Topology-imbalance by Relieving Under-reaching and Over-squashing

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paper: <https://arxiv.org/pdf/2208.08302.pdf>

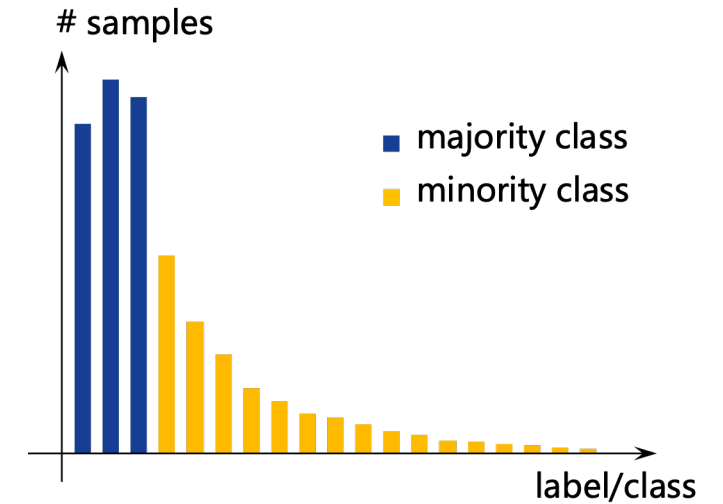
code: <https://github.com/RingBDStack/PASTEL>

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■ *Imbalance Learning*

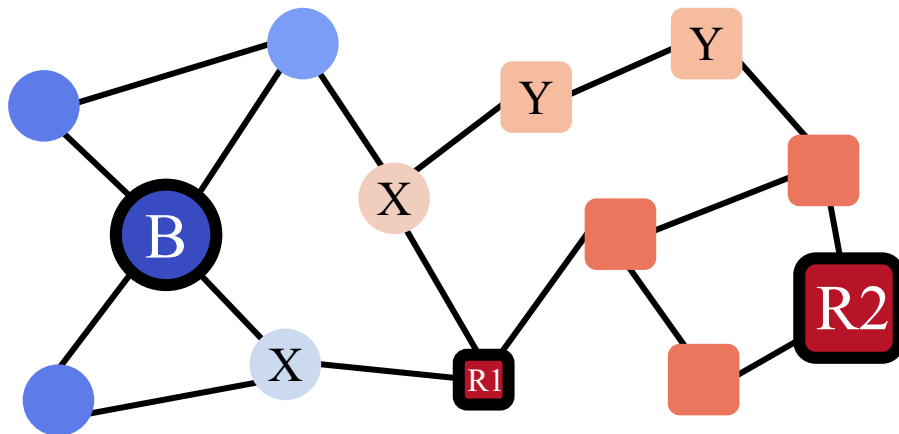
■ Class-imbalance (Quantity-imbalance)

- performance is dominated by the **majority class**
- **Re-sampling**: balance the number (*data-level*)
- **Re-weighting**: adjust sample weights (*algorithm-level*)



■ **Topology-imbalance** (graph-specific, node classification)

- asymmetric & uneven topology positions of **labeled nodes**



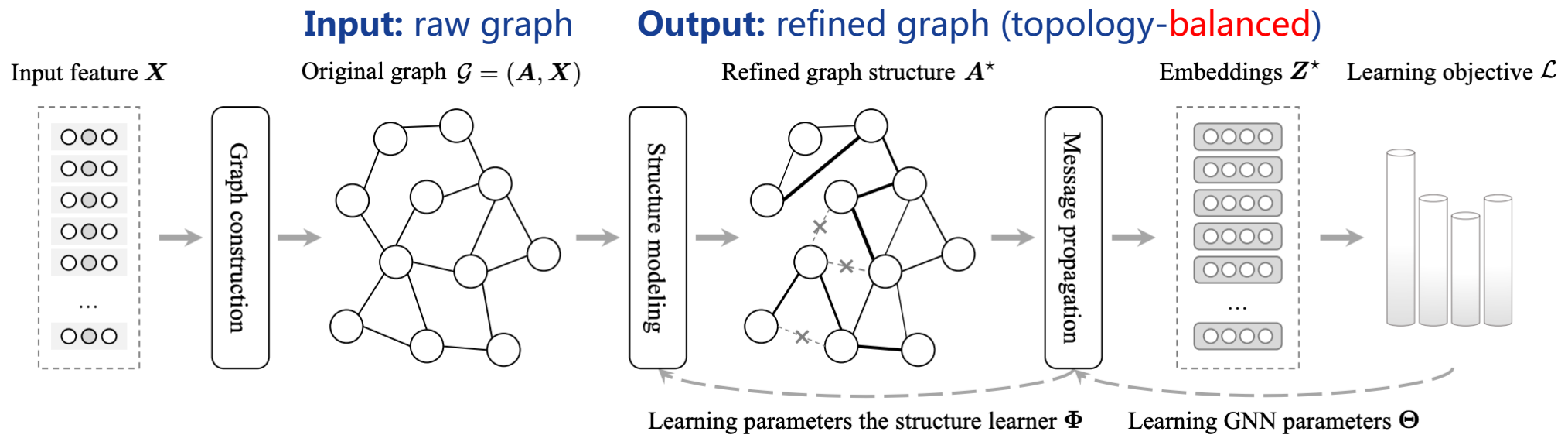
■ ReNode

- classical **re-weighting on node loss**
- directed by an influence conflict detection based metric
- Other method? **Graph Structure Learning!**

■ Graph Structure Learning (GSL)

■ **Target:** jointly learning an optimized **structure** and **corresponding representations** to improving the robustness of GNN models.

■ **Paradigm: Structure Modeling** → **Message Passing** → **Learning Objective**

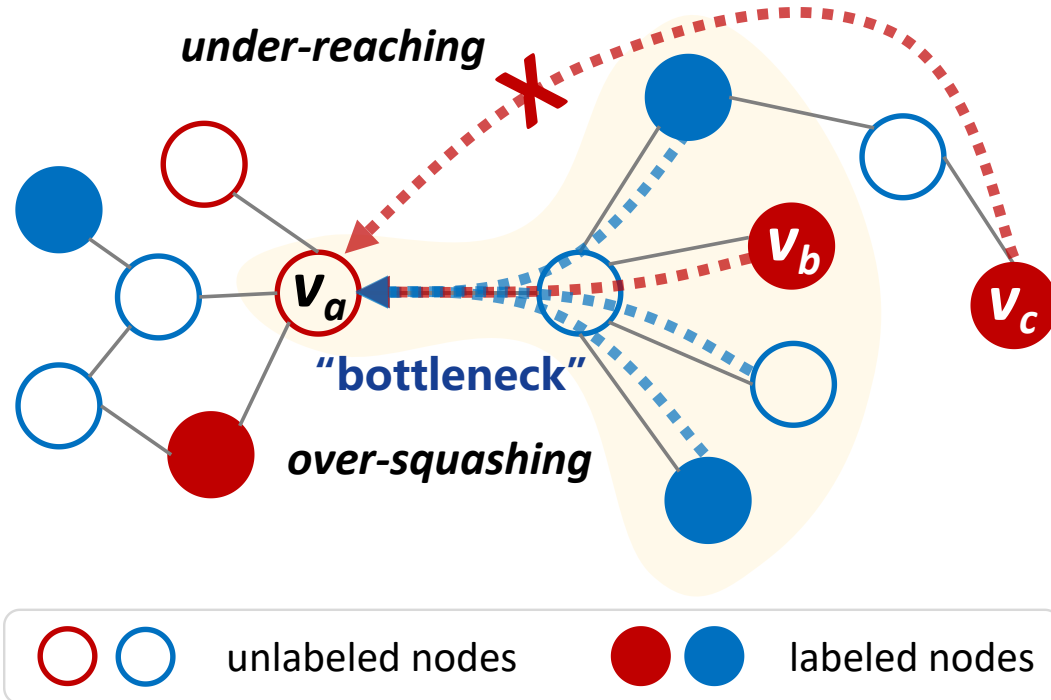


■ Measurements for modeling nodes relations

■ node features/degrees/encodings, edge attributes, etc. **Node positions?**

■ Understanding Topology-imbalance

- Q1: Why does topology-imbalance affect graph representation learning?
- Q2: What kind of graphs are susceptible to topology-imbalance?



- Under-reaching → Reaching Coefficient (*RC*)

$$RC = \frac{1}{|\mathcal{V}_U|} \sum_{v_i \in \mathcal{V}_U} \frac{1}{|\mathcal{V}_L^{y_i}|} \sum_{v_j \in \mathcal{V}_L^{y_i}} \left(1 - \frac{\log |\mathcal{P}_{sp}(v_i, v_j)|}{\log D_{\mathcal{G}}} \right)$$

RC ↑: better reachability (more shortcuts/paths)

- Over-squashing → Squashing Coefficient (*SC*)

$$SC = \frac{1}{|\mathcal{V}_U|} \sum_{v_i \in \mathcal{V}_U} \frac{1}{|\mathcal{N}_{y_i}(v_i)|} \sum_{v_j \in \mathcal{N}_{y_i}(v_i)} \frac{\sum_{e_{kt} \in \mathcal{P}_{sp}(v_i, v_j)} Ric(v_k, v_t)}{|\mathcal{P}_{sp}(v_i, v_j)|}$$

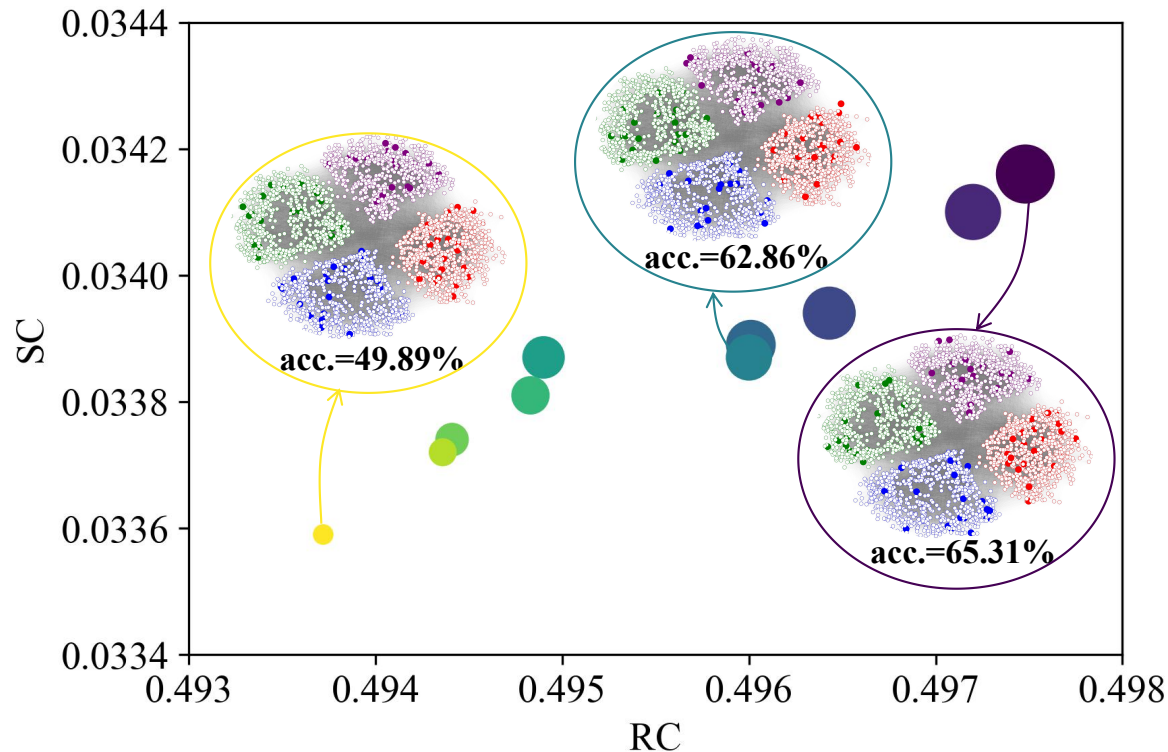
SC ↑: lower squashing (more ring structures)

- **Conclusion: hurts the performance by under-reaching and over-squashing.**

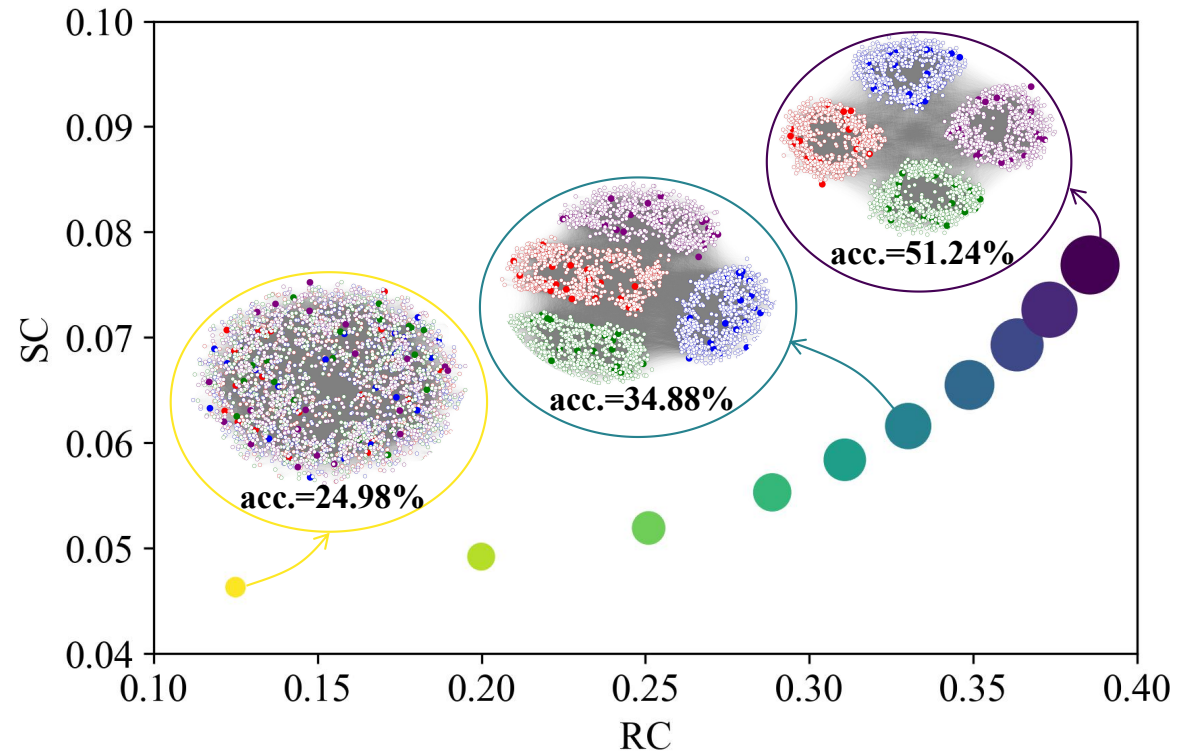
■ *Understanding Topology-imbalance*

- Q1: Why does topology-imbalance affect graph representation learning?
- Q2: What kind of graphs are susceptible to topology-imbalance? SBM $\mathcal{G}(N, C, p, q)$

same structure + **different** labeled nodes

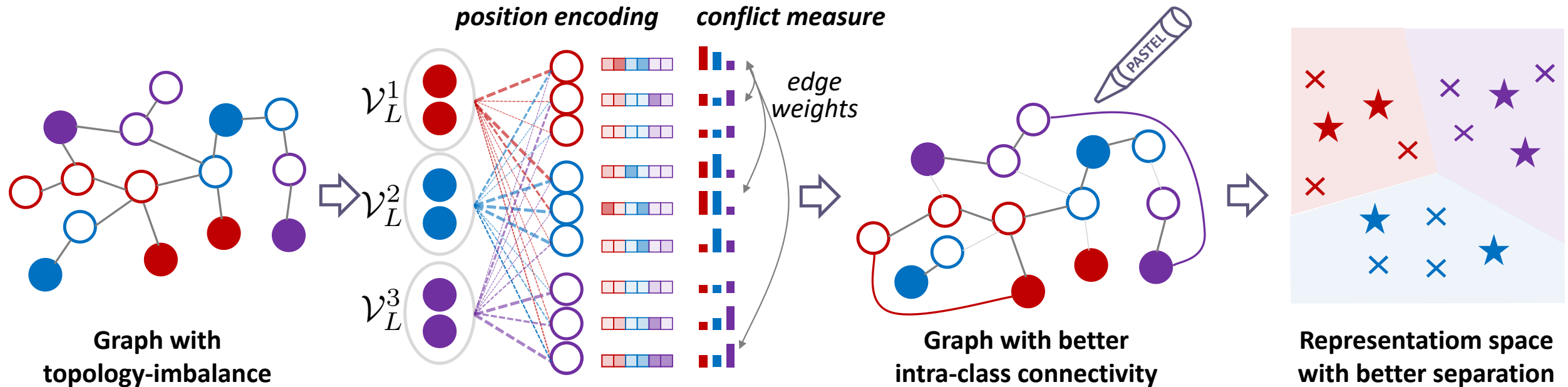


different structure + **same** labeled nodes



■ **Conclusion: poor reachability (smaller *RC*) and stronger squashing (smaller *SC*)**

Position-Aware Structure Learning framework (PASTEL)



- **Task:** Semi-supervised node classification
- **Position-aware Structure Learning:** anchor-based position encoding method
- **Class-wise Conflict Measure:** guide what nodes should be more closely connected
- **Learning with the Optimized Structure:** original + position + feature + constraints

Position-aware Structure Learning

Anchor-based Position Encoding

separate labeled nodes **by class**: $\mathcal{V}_L = \{\mathcal{V}_L^1, \mathcal{V}_L^2, \dots, \mathcal{V}_L^C\}$

considerate an **unlabeled node** v_i (e.g. v_6):

measure **position relations** between v_i and anchor sets:

$$\phi(v_i, \mathcal{V}_L^c) = \frac{\sum_{v_j \in \mathcal{N}_c(v_i)} |\mathcal{P}_{sp}(v_i, v_j)|}{|\mathcal{N}_c(v_i)|}$$

position-aware encoding of node v_i :

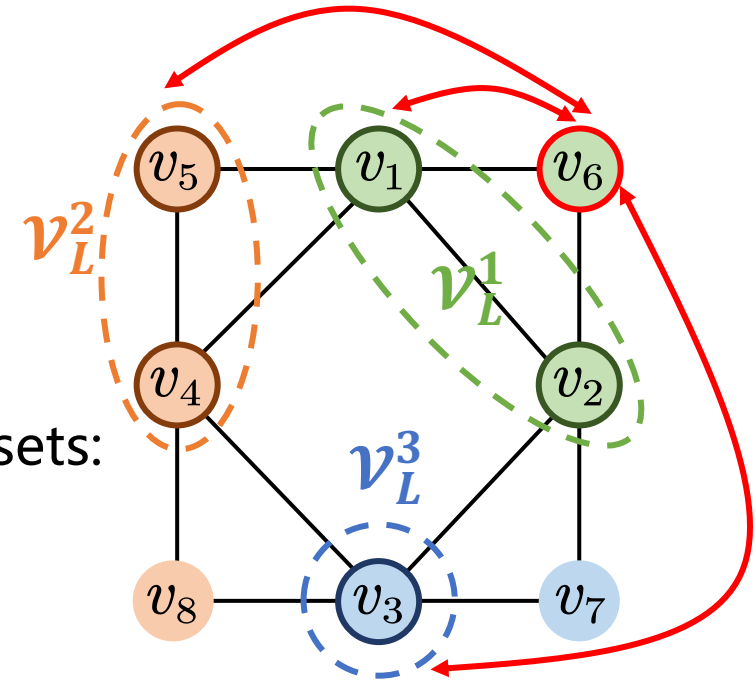
$$\mathbf{p}_i = \left(\phi(v_i, \mathcal{V}_L^1), \phi(v_i, \mathcal{V}_L^2), \dots, \phi(v_i, \mathcal{V}_L^C) \right)$$

transform to **learnable** vector: $\mathbf{h}_i^p = \mathbf{W}_\phi \cdot \mathbf{p}_i$

Position-aware Metric Learning

consider both **feature info** and **position-based similarity** to form an edge:

$$a_{ij}^p = \frac{1}{m} \sum_{h=1}^m \cos \left(\mathbf{W}_h \cdot (z_i || \mathbf{h}_i^p), \mathbf{W}_h \cdot (z_j || \mathbf{h}_j^p) \right)$$



$$\mathcal{V}_L = \{\mathcal{V}_L^1, \mathcal{V}_L^2, \mathcal{V}_L^3\}$$

$$\mathbf{p}_6 = \left(\phi(v_6, \mathcal{V}_L^1), \phi(v_6, \mathcal{V}_L^2), \phi(v_6, \mathcal{V}_L^3) \right)$$

$$\mathbf{h}_6^p = \mathbf{W}_\phi \cdot \mathbf{p}_6$$

■ *Class-wise Conflict Measure*

■ Group PageRank (GPR)

- traditional PageRank → **Group** PageRank (label-aware):

measure supervision information **from labeled nodes of each class**

$$\mathbf{P}^{gpr}(c) = (1 - \alpha)\mathbf{A}'\mathbf{P}^{gpr}(c) + \alpha\mathbf{I}_c$$

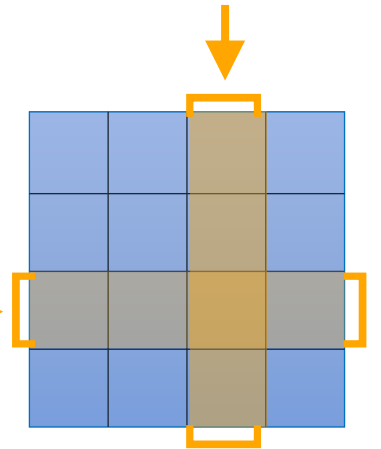
$$\mathbf{P}^{gpr} = \alpha (\mathbf{E} - (1 - \alpha)\mathbf{A}')^{-1} \mathbf{I}^*$$

$$\mathbf{I}_c^i = \begin{cases} \frac{1}{|\mathcal{V}_L^c|}, & \text{if } y_i = c \\ 0, & \text{otherwise} \end{cases}$$

$$\mathbf{I}^* = \{\mathbf{I}_c, c = 1, 2, \dots, C\}$$

i-th row
GPR vector of v_i

c-th dimension
influence of labeled nodes
of class c on v_i



- **Expecting:** GPR vector of nodes to form a **“sharp”** distribution focusing on their **ground truth label** (see Experiments Sec.)

■ Control the **connection strength** of an edge

- GPR vectors measure **conflict**: $\kappa_{ij} = \text{KL}(\mathbf{P}_i^{gpr}, \mathbf{P}_j^{gpr})$

- conflict → edge weight: $w_{ij} = \frac{1}{2} \left[-\cos \frac{\text{Rank}(\kappa_{ij})}{|\mathcal{V}| \times |\mathcal{V}|} * \pi + 1 \right]$, $\tilde{a}_{ij}^P = w_{ij} \cdot a_{ij}^P \rightarrow$ position-aware \mathbf{A}_P

■ Learning with the Optimized Structure

■ Graph structure mixing and optimization

■ **position-aware** adjacency: $A_P = \{\tilde{a}_{ij}^P, i, j \in \{1, 2, \dots, N\}\}$

■ **node feature view** adjacency: $A_N = \{a_{ij}^N, i, j \in \{1, 2, \dots, N\}\},$

$$a_{ij}^N = \frac{1}{m} \sum_{h=1}^m \cos \left(\mathbf{W}_h \cdot \left(\mathbf{x}_i \parallel \mathbf{h}_i^{p_0} \right), \mathbf{W}_h \cdot \left(\mathbf{x}_j \parallel \mathbf{h}_j^{p_0} \right) \right)$$

■ **mixing:**

$$\mathbf{A}^* = \lambda_1 \mathbf{D}^{-\frac{1}{2}} \mathbf{A} \mathbf{D}^{-\frac{1}{2}} + (1 - \lambda_1) (\lambda_2 f(\mathbf{A}_N) + (1 - \lambda_2) f(\mathbf{A}_P)) \rightarrow \boxed{\mathcal{G}^* = (\mathbf{A}^*, \mathbf{X})}$$

■ Learning objectives

■ structure quality control: $\mathcal{L}_{smooth} = \frac{1}{N^2} \text{tr} \left(\mathbf{X}^T \mathbf{L}^* \mathbf{X} \right), \mathcal{L}_{con} = \frac{1}{N} \mathbf{1}^T \log(\mathbf{A}^* \mathbf{1}), \mathcal{L}_{spar} = \frac{1}{N^2} \|\mathbf{A}^*\|_F^2$
smooth *connectivity* *sparsity*

■ classification loss: $\mathbf{Z} = \text{GNN-Encoder}(\mathbf{A}^*, \mathbf{X}), \hat{\mathcal{Y}} = \text{Classifier}(\mathbf{Z}), \mathcal{L}_{cls} = \text{Cross-Entropy}(\mathcal{Y}, \hat{\mathcal{Y}})$

■ **overall loss:** $\mathcal{L} = \mathcal{L}_{cls} + \beta_1 \mathcal{L}_{smooth} + \beta_2 \mathcal{L}_{con} + \beta_3 \mathcal{L}_{spar}.$

■ *Experimental Setups*

■ Datasets

- **real-world datasets:** Cora, Citeseer, Photo, Actor, Chameleon, Squirrel

- **synthetic graph:** Stochastic Block Model (SBM)

■ Baselines

- **GNN backbones:** GCN, GAT, APPNP, GraphSAGE

- **topology-imbalance specific baselines:** ReNode

- **graph structure learning baselines:** DropEdge, AddEdge, SDRF, NeuralSparse, IDGL

- **Classification task setting:** set the number of **labeled nodes** in each class to be **20**

■ Metrics

- Weighted-F1 (W-F1) \pm standard deviation

- Macro-F1 (M-F1) \pm standard deviation

Node Classification on Real-world Graphs

Backbone	Model	Cora		Citeseer		Photo		Actor		Chameleon		Squirrel	
		W-F1	M-F1	W-F1	M-F1	W-F1	M-F1	W-F1	M-F1	W-F1	M-F1	W-F1	M-F1
GCN	original	79.4±0.9	77.5±1.5	66.3±1.3	62.2±1.2	85.4±2.8	84.6±1.3	21.8±1.3	20.9±1.4	30.5±3.4	30.5±3.3	21.9±1.2	21.9±1.2
	ReNode	80.0±0.7	78.4±1.3	66.4±1.0	62.4±1.1	86.2±2.4	85.3±1.6	21.2±1.2	20.2±1.6	30.3±3.2	30.4±2.8	22.4±1.1	22.4±1.1
	AddEdge	79.0±0.9	77.0±1.4	66.2±1.3	62.2±1.3	85.5±1.5	86.1±1.8	21.2±1.3	20.3±1.5	30.6±1.6	30.4±1.7	21.7±1.5	21.7±1.5
	DropEdge	79.8±0.8	77.8±1.0	66.6±1.4	63.4±1.6	86.8±1.7	85.4±1.3	22.4±1.0	21.4±1.3	30.6±3.5	30.6±3.3	22.8±1.2	22.8±1.2
	SDRF	82.1±0.8	80.6±0.8	69.6±0.4	66.6±0.3	> 5 days	> 5 days	> 5 days	> 5 days	39.1±1.2	39.0±1.2	> 5 days	> 5 days
	NeuralSparse	81.7±1.4	80.9±1.4	71.8±1.2	69.0±1.0	89.7±1.9	88.7±1.8	24.4±1.5	23.6±1.6	44.9±3.0	44.9±2.8	28.1±1.8	28.1±1.8
	IDGL	82.3±0.6	81.0±0.9	71.7±1.0	68.0±1.3	88.6±2.3	88.8±1.4	24.9±0.8	22.0±0.7	55.4±1.8	55.0±1.7	28.8±2.3	28.9±2.2
	PASTEL	82.5±0.3	81.2±0.3	72.9±0.8	69.3±0.9	91.4±2.7	91.3±2.2	26.4±1.0	24.4±1.2	57.8±2.4	57.3±2.4	37.5±0.6	37.5±0.7
GAT	original	78.3±1.5	76.4±1.7	64.4±1.7	60.6±1.7	88.2±2.9	86.2±2.6	21.8±1.2	20.9±1.1	29.9±3.5	29.9±3.1	20.5±1.4	20.5±1.4
	ReNode	78.9±1.2	77.2±1.5	64.9±1.6	61.0±1.5	89.1±2.4	87.1±2.6	21.5±1.2	20.5±1.1	29.2±2.3	29.1±2.0	20.4±1.8	20.4±1.8
	AddEdge	78.0±1.6	76.2±1.6	64.0±1.3	60.2±1.3	88.2±2.4	86.2±2.5	21.3±1.2	20.3±1.1	29.8±1.7	29.6±1.5	20.7±1.6	20.7±1.6
	DropEdge	78.7±1.3	76.9±1.5	64.5±1.4	60.5±1.3	88.9±1.9	87.1±2.1	22.9±1.2	21.8±1.1	30.3±1.6	30.2±1.2	21.2±1.5	21.2±1.5
	SDRF	77.9±0.7	75.9±0.9	64.9±0.6	61.9±0.9	> 5 days	> 5 days	> 5 days	> 5 days	43.0±1.9	42.5±1.9	> 5 days	> 5 days
	NeuralSparse	81.4±4.8	79.4±4.8	64.8±1.5	61.9±1.3	90.2±2.5	88.0±2.3	23.4±1.7	22.4±1.5	45.6±2.1	45.5±1.8	28.8±1.3	28.8±1.3
	IDGL	80.6±1.0	79.7±0.9	66.5±1.5	61.9±1.9	89.9±3.1	87.7±2.6	22.4±1.5	21.8±1.2	48.4±4.0	47.8±3.1	27.0±2.6	27.0±2.6
	PASTEL	81.9±1.4	80.7±1.2	66.6±1.9	62.0±1.7	91.8±3.2	89.4±2.9	24.4±2.6	22.1±2.6	52.1±2.7	52.5±2.8	35.3±0.9	35.3±0.8
APPNP	original	80.6±1.6	79.3±1.2	66.5±1.5	62.3±1.5	89.3±1.6	86.3±1.7	21.1±1.5	20.7±1.1	35.3±4.0	35.0±3.8	23.1±1.6	23.1±1.6
	ReNode	81.1±0.9	79.9±0.9	66.6±1.7	62.4±1.6	89.6±1.4	87.2±1.3	20.2±2.0	20.0±1.7	33.5±2.5	33.3±2.3	23.9±2.0	23.9±2.0
	AddEdge	80.3±1.3	78.8±1.1	66.6±2.1	62.5±2.1	89.3±1.2	86.4±1.2	21.5±1.3	20.7±1.4	35.7±1.7	35.4±1.2	23.1±1.6	23.2±1.7
	DropEdge	80.9±1.4	79.4±1.2	66.7±2.0	63.0±1.9	90.0±1.2	87.0±1.2	21.8±1.8	20.8±1.4	36.0±1.7	35.7±1.6	23.3±1.7	23.3±1.7
	SDRF	80.7±0.9	79.1±0.8	67.1±0.6	63.1±0.8	> 5 days	> 5 days	> 5 days	> 5 days	36.5±2.1	35.8±2.1	> 5 days	> 5 days
	NeuralSparse	81.1±1.4	79.9±1.2	66.8±1.9	62.7±1.9	91.3±1.8	89.4±1.6	21.8±1.9	21.4±1.5	39.1±2.9	38.7±2.8	28.3±1.5	28.3±1.5
	IDGL	81.3±0.9	80.2±0.9	67.0±1.3	62.9±1.3	91.6±1.3	88.6±2.2	21.4±2.4	20.1±2.4	41.2±2.2	40.6±2.6	29.6±2.3	29.7±2.2
	PASTEL	82.0±1.0	80.0±0.9	67.3±1.3	63.2±1.5	92.3±3.1	89.9±2.5	22.5±2.0	20.9±2.1	44.2±3.2	43.8±3.4	34.6±1.6	34.6±1.6
GraphSAGE	original	75.4±1.6	74.1±1.6	64.8±1.6	60.7±1.6	86.1±2.5	83.3±2.4	24.0±1.2	23.2±1.0	36.5±1.6	36.2±1.6	27.2±1.7	27.2±1.7
	ReNode	76.4±0.9	75.0±1.1	65.4±1.7	61.2±1.7	86.5±1.7	84.1±1.7	23.7±1.2	22.8±1.0	36.4±1.9	36.1±1.9	27.7±1.8	27.7±1.8
	AddEdge	75.2±1.2	73.7±1.2	65.0±1.4	60.9±1.3	86.1±2.8	83.4±2.6	23.8±1.7	23.2±1.6	36.5±1.5	36.2±1.3	26.9±2.1	26.9±2.1
	DropEdge	76.0±1.6	74.5±1.6	65.1±1.4	60.9±1.4	86.2±1.6	83.5±1.4	24.1±1.0	23.3±0.9	37.5±1.4	37.2±1.4	27.5±1.8	27.5±1.8
	SDRF	75.7±0.8	74.6±0.8	65.3±0.6	61.4±0.6	> 5 days	> 5 days	> 5 days	> 5 days	41.5±2.6	41.6±2.7	> 5 days	> 5 days
	NeuralSparse	79.7±1.8	77.8±1.6	64.7±1.4	61.1±1.3	89.1±5.4	86.7±5.5	25.1±1.2	24.4±1.1	39.1±1.9	39.0±1.9	32.2±2.4	32.2±2.4
	IDGL	79.2±0.9	78.4±0.8	65.6±0.9	61.3±1.2	90.0±1.0	86.3±1.3	24.0±2.6	22.4±2.7	43.8±3.4	43.0±3.2	33.9±0.9	33.9±0.8
	PASTEL	81.1±0.8	79.8±0.7	65.7±1.1	61.4±1.4	92.0±0.6	89.0±1.0	26.0±2.4	23.6±2.7	47.7±0.9	46.9±0.9	35.5±1.4	35.5±1.4

■ **PASTEL shows overwhelming superiority in improving the performance of backbones on all datasets.**

■ *Node Classification on Cora with different imbalance level*

- Cora-L, Cora-M, Cora-H: topology-imbalance level low(L)/medium(M)/high(H)
- GNN backbone: GCN

	Cora-L		Cora-M		Cora-H	
	RC	SC	RC	SC	RC	SC
	0.4130	-0.6183	0.4100	-0.6204	0.4060	-0.6302
	W-F1 (%)	Δ (%)	W-F1 (%)	Δ (%)	W-F1 (%)	Δ (%)
GCN	80.9±0.9	—	78.8±0.8	—	77.5±1.0	—
ReNode	81.3±0.7	↑0.4	79.3±0.8	↑0.5	78.3±1.1	↑0.8
SDRF	81.0±0.7	↑0.1	78.9±0.8	↑0.1	77.9±0.7	↑0.4
IDGL	82.5±1.0	↑1.6	80.4±1.0	↑1.6	81.6±1.1	↑4.1
PASTE L	82.7±0.9	↑1.8	81.0±0.9	↑2.2	81.9±1.1	↑4.4

- **PASTE**L performs best on all datasets with different imbalance level.
- achieve up to **4.4%** improvement on the **highly topology-imbalance** dataset.

■ *Node Classification on Synthetic Graphs*

- **Stochastic Block Model (SBM) $\mathcal{G}(N, C, p, q)$ ($N=3000, C=6$)**
- **GNN backbone: GCN**

topology-imbalance level: high \rightarrow low

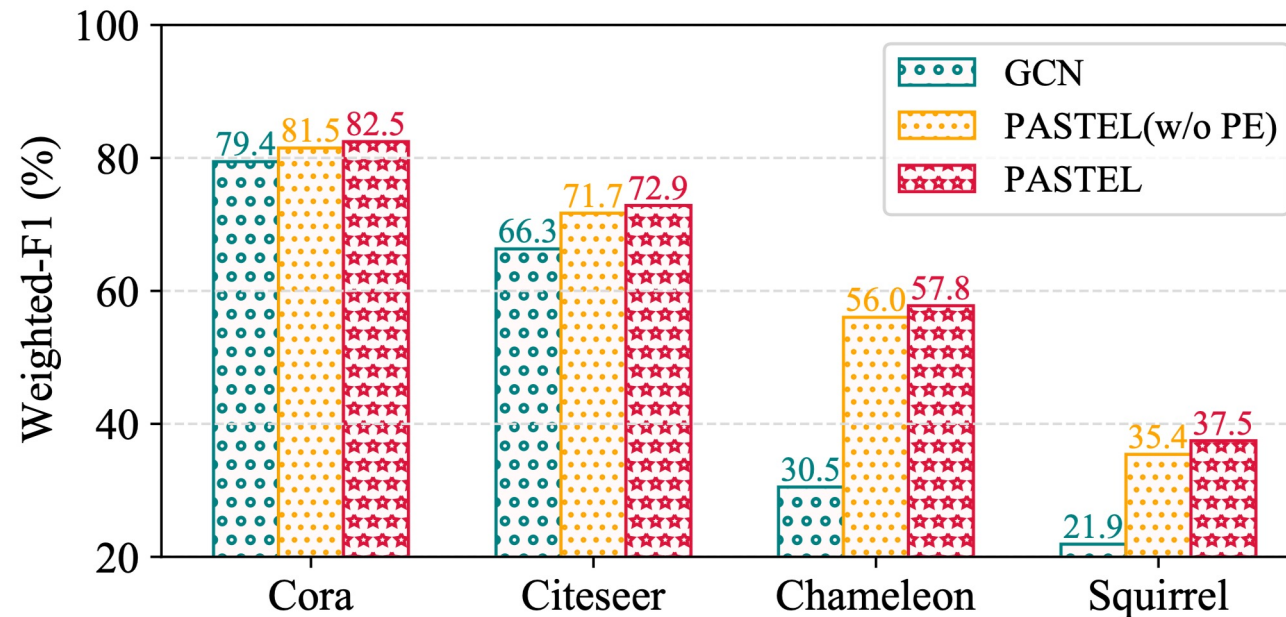
	SBM-1		SBM-2		SBM-3		SBM-4		SBM-5		SBM-6		SBM-7	
p	0.5000		0.5000		0.5000		0.5000		0.5000		0.5000		0.5000	
q	0.0300		0.0100		0.0083		0.0071		0.0063		0.0056		0.0050	
RC	0.4979		0.4984		0.4990		0.4994		0.5002		0.5004		0.5009	
SC	0.0998		0.0999		0.1000		0.1001		0.1007		0.1017		0.1144	
	W-F1	Δ	W-F1	Δ	W-F1	Δ	W-F1	Δ	W-F1	Δ	W-F1	Δ	W-F1	Δ
GCN	40.29	—	42.37	—	42.99	—	44.13	—	45.19	—	45.21	—	45.22	—
ReNode	41.33	$\uparrow 1.04$	42.40	$\uparrow 0.03$	43.21	$\uparrow 0.22$	44.56	$\uparrow 0.43$	45.20	$\uparrow 0.01$	45.08	$\downarrow 0.13$	44.89	$\downarrow 0.33$
PASTEL	45.67	$\uparrow 5.38$	57.61	$\uparrow 15.24$	58.33	$\uparrow 15.34$	60.29	$\uparrow 16.16$	66.41	$\uparrow 21.22$	66.45	$\uparrow 21.24$	66.57	$\uparrow 21.35$

- **PASTEL can increase the classification Weighted-F1 score by **5.38%-21.35%** on SBM graphs **with different community structures**, showing superior effectiveness.**

■ Ablation Study: Impact of the Position Encoding

■ Variant: PASTEL (w/o PE) with GCN as backbone

- removes the position encoding module
- directly take the node features for metric learning



■ The **structure learning strategy** of PASTEL contributes the most.

■ The **position encoding** still benefits learning better structure.

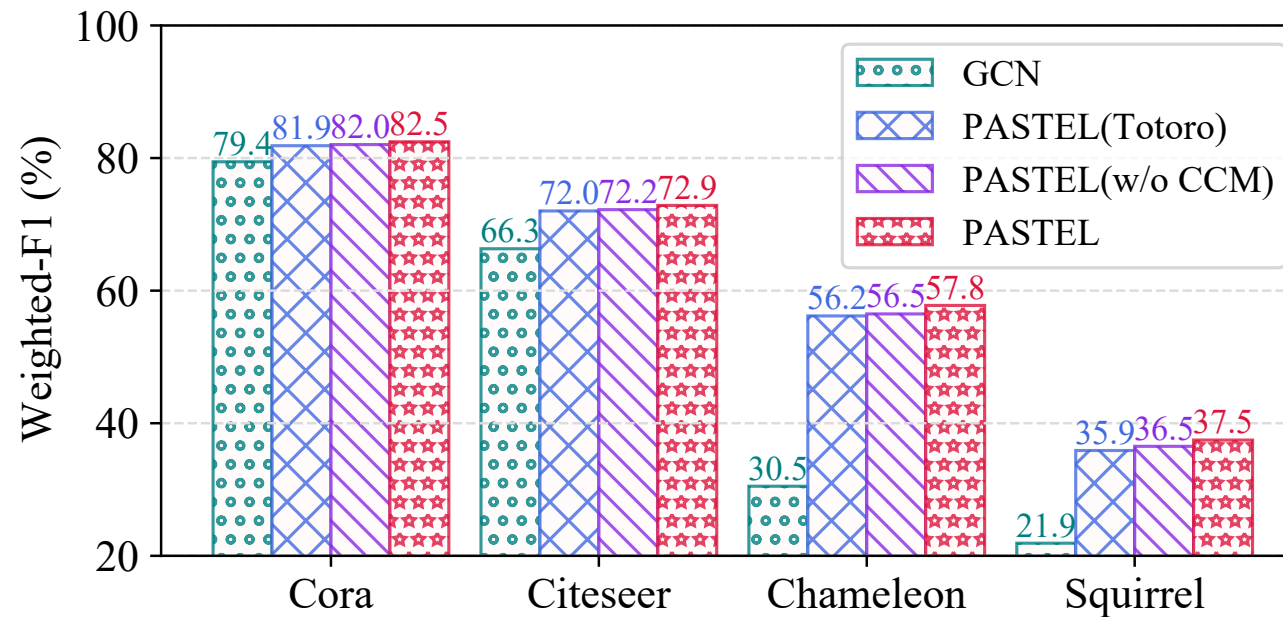
■ Ablation Study: Impact of the Class-wise Conflict Measure

■ Variant: PASTEL (w/o CCM) with GCN as backbone

- remove the class-wise conflict measure, directly takes the learned edge possibilities

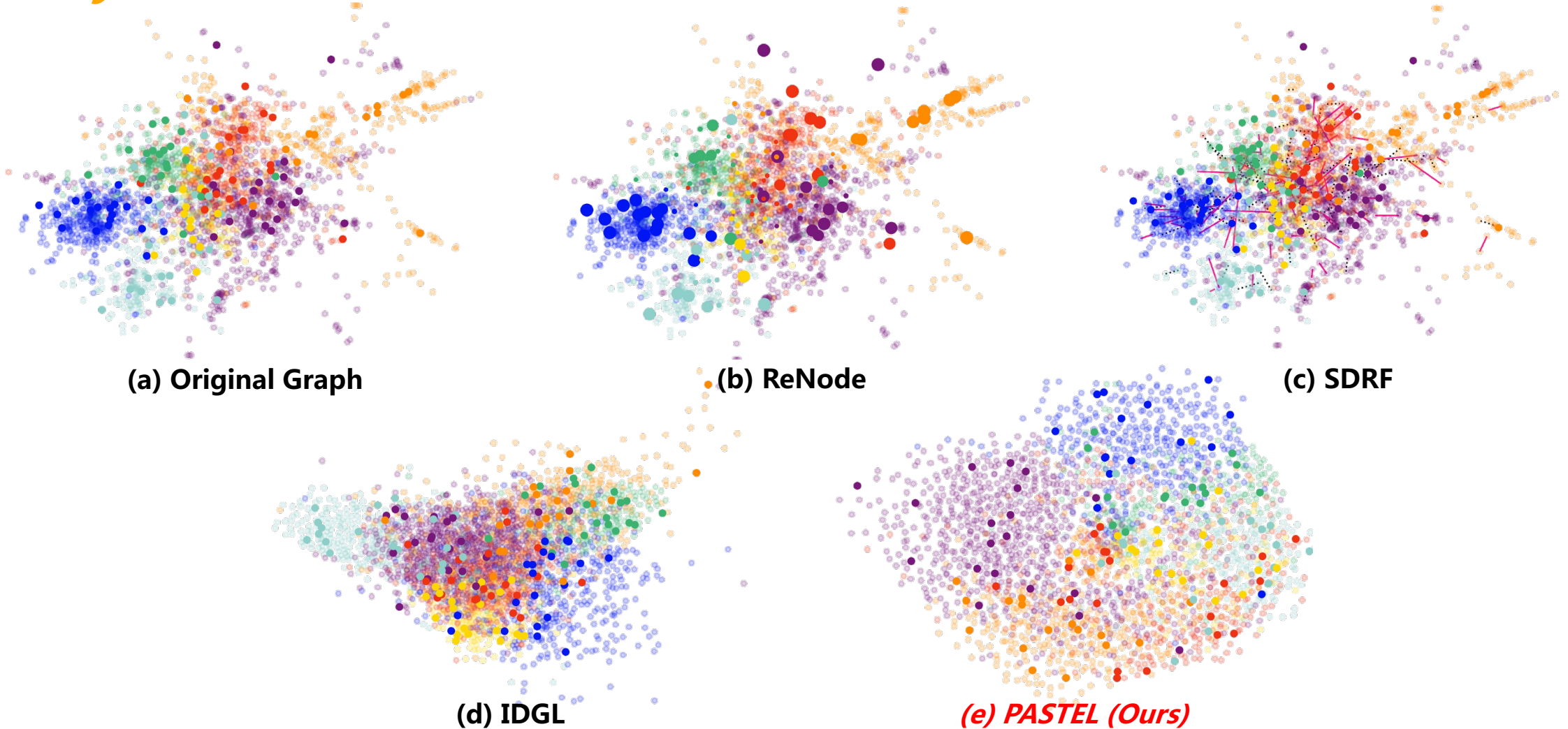
■ Variant: PASTEL (Totoro) with GCN as backbone

- take the Totoro metric introduced in ReNode as the conflict measure of nodes



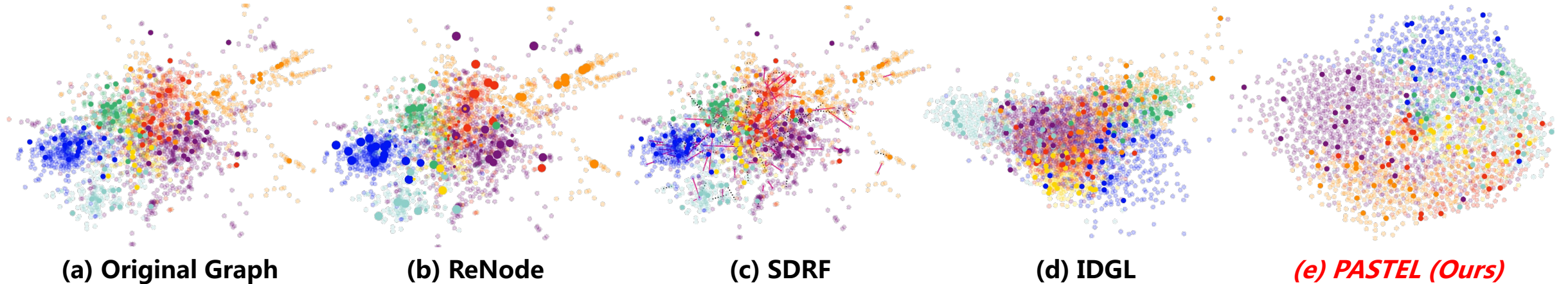
■ PASTEL consistently outperforms the other two variants.

■ Analysis of Learned Structure: Visualization of Cora



■ PASTEL can obtain graph structure with **clearer class boundaries**.

■ Analysis of Learned Structure: Change of RC and SC

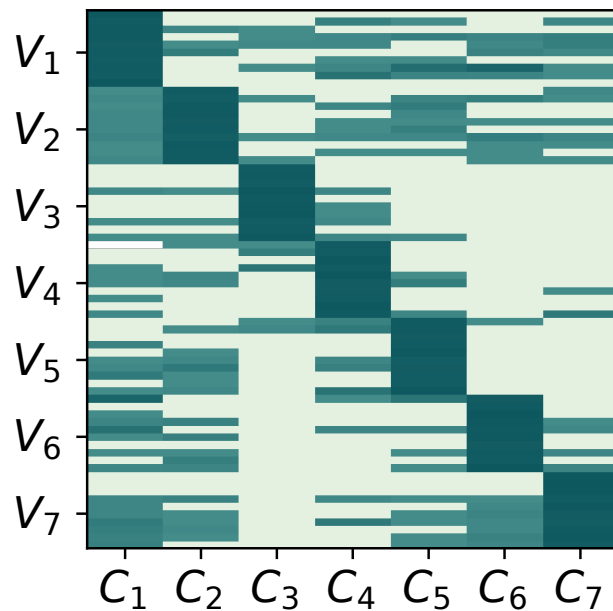


	Original Graph	ReNode	SDRF	IDGL	PASTEL
<i>RC</i>	0.4022	0.4022	0.4686	0.5028	0.5475
<i>SC</i>	-0.6299	-0.6299	-0.4942	-0.4069	-0.3389
W-F1 (%)	79.44	80.34	82.01	82.38	82.86

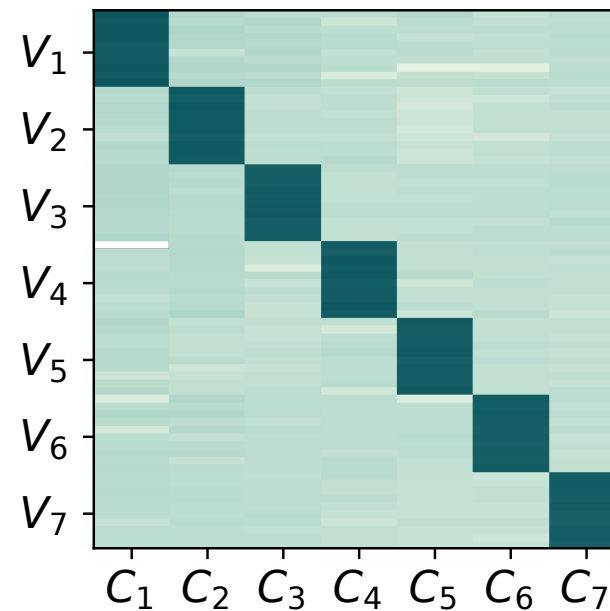
■ All the **structure learning methods** learn structures with larger RC & SC, leading the performance improvement of node classification.

■ Analysis of Learned Structure: Change of GPR Vector

- randomly choose **10** nodes for **each class** in Cora
- visualize their GPR vectors on **the original graph** and **the learned graph**



(a) Original Graph



(b) Learned Graph

- V_i : 10 nodes of class i
- C_i : the i -th class

- the class-wise conflict measure plays an important role on giving guidance for **more class connectivity orthogonality**.

■ *Conclusion and Future Works*

- We provide **a new understanding** and **two metrics** of topology-imbalance in the perspective of **under-reaching** and **over-squashing**, answering:
 - how topology-imbalance affects GNN' s performance
 - what graphs are susceptible to topology-imbalance
- PASTEL designs an **anchor-based position encoding mechanism** and a **class-wise conflict measure** to obtain structures with **better in-class connectivity**.
- Future works: incorporate the proposed two quantitative metrics **into the learning process** to address topology-imbalance more directly.

Position-aware Structure Learning for Graph Topology-imbalance by Relieving Under-reaching and Over-squashing

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